AN INFINITE FAMILY OF GROMOLL-MEYER SPHERES

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Abstract. We construct a new infinite family of models of exotic 7-spheres. These models are direct generalizations of the Gromoll-Meyer sphere. From their symmetries, geodesics and submanifolds half of them are closer to the standard 7-sphere than any other known model for an exotic 7-sphere.

1. Introduction

This paper provides a new geometric way to construct all exotic 7-spheres. Exotic spheres are differentiable manifolds that are homeomorphic but not diffeomorphic to standard spheres. The first examples were found by Milnor [Mi1] in 1956 among the $\mathbb{S}^3$-bundles over $\mathbb{S}^4$. It turned out that 7 is the smallest dimension where exotic spheres can occur except possibly in the special dimension 4. In any dimension $n > 4$ the exotic spheres and the standard sphere form a finite abelian group: the group Θ_n of (orientation preserving diffeomorphism classes of) homotopy spheres [KM]. The inverse element in Θ_n can be obtained by a change of orientation. In dimension 7 we have $\Theta_7 \approx \mathbb{Z}_{28}$. Hence, ignoring orientation there are 14 exotic 7-spheres. From these 14 exotic 7-spheres four (corresponding to $2, 5, 9, 12, 16, 19, 23, 26 \in \mathbb{Z}_{28}$) are not diffeomorphic to an $\mathbb{S}^3$-bundle over $\mathbb{S}^4$ [EK].

In 1974 Gromoll and Meyer [GM] constructed an exotic 7-sphere, Σ^7_{GM}, as quotient of the compact group $\mathrm{Sp}(2)$ by a two-sided $\mathbb{S}^3$-action. This construction provided Σ^7_{GM} automatically with a metric of nonnegative sectional curvature ($K \geq 0$). The Gromoll-Meyer sphere Σ^7_{GM} was the only exotic sphere known to admit such a metric until 1999 when Grove and Ziller [GZ] constructed metrics with $K \geq 0$ on all Milnor spheres, i.e., on all exotic 7-spheres that are $\mathbb{S}^3$-bundles over $\mathbb{S}^4$. In 2002 Totaro [To] and independently Kapovitch and Ziller [KZ] showed that Σ^7_{GM} is the only exotic sphere that can be modeled by a biquotient of a compact group and thus underlined the singular status of the Gromoll-Meyer sphere among all models for exotic spheres.

We nevertheless provide an elementary and direct generalization of the Gromoll-Meyer construction. The essential components in this construction are natural self-maps of $\mathbb{S}^7$, namely, the n-powers of unit octonions, $n \in \mathbb{Z}$. In terms of quaternions these maps are defined by

$$\rho_n : \mathbb{S}^7 \to \mathbb{S}^7, \quad \left(\begin{smallmatrix} \cos t + p \sin t \\ w \sin t \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} \cos nt + p \sin nt \\ w \sin nt \end{smallmatrix}\right)$$

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where $p \in \operatorname{Im}\mathbb{H}$ and $w \in \mathbb{H}$ with $|p|^2 + |w|^2 = 1$. Let $\langle\langle u, v\rangle\rangle := \bar{u}^{\mathrm{t}} v$ denote the standard Hermitian product on $\mathbb{H}^2$. The submanifolds

$$E_n^{10} := \{(u, v) \in \mathbb{S}^7 \times \mathbb{S}^7 \mid \langle\langle \rho_n(u), v\rangle\rangle = 0\}$$

come equipped with a free action of the unit quaternions:

$$\mathbb{S}^3 \times E_n^{10} \to E_n^{10}, \quad q \star (u, v) = (qu\bar{q}, qv).$$

Here, $qu\bar{q}$ means that the two quaternionic components of u are simultaneously conjugated by $q \in \mathbb{S}^3$. The quotient of E_n^{10} by the free $\star$-action is a smooth manifold

$$\Sigma_n^7 := E_n^{10}/\mathbb{S}^3.$$

For $n = 1$ we have $E_1^{10} = \mathrm{Sp}(2)$ (the group of quaternionic 2×2 matrices A with $\bar{A}^{\mathrm{t}} A = \mathbb{1}$) and the $\star$-action is the original Gromoll-Meyer action. Hence, $\Sigma_1^7 = \Sigma_{\mathrm{GM}}^7$. It is also easy to see that Σ_0^7 is diffeomorphic to $\mathbb{S}^7$.

Theorem 1. *The differentiable manifold Σ_n^7 is a homotopy sphere and represents the (n mod 28)-th element in $\Theta_7 \approx \mathbb{Z}_{28}$.*

Let $\mathbb{Z}_2 \times \mathbb{Z}_2$ denote the diagonal matrices of $\mathrm{O}(2) \subset \mathrm{Sp}(2)$. All E_n^{10} admit a smooth action of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{S}^3$ that commutes with the free $\star$-action:

$$\mathbb{Z}_2 \times \mathbb{Z}_2 \times E_n^{10} \to E_n^{10}, \quad B \bullet (u, v) = (Bu, Bv),$$
$$\mathbb{S}^3 \times E_n^{10} \to E_n^{10}, \quad q \bullet (u, v) = (u, v\bar{q}).$$

The induced effective action on Σ_n^7 is an action of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathrm{SO}(3)$ where $\mathrm{SO}(3) = \mathbb{S}^3/\{\pm 1\}$. On Σ_0^7 this action can be identified with the linear action

$$(B, \pm q) \cdot (x, u) = (Bx, Bqu\bar{q})$$

on $\mathbb{S}^7 \subset \mathbb{R}^2 \times (\operatorname{Im}\mathbb{H})^2$. On $\Sigma_1^7 = \Sigma_{\mathrm{GM}}^7$ the action coincides with the subaction of the $\mathrm{O}(2) \times \mathrm{SO}(3)$-action given in [GM].

The surprising fact is the following even/odd grading of the Σ_n^7:

Theorem 2. *All Σ_n^7 with even n are equivariantly homeomorphic to $\mathbb{S}^7$ with the linear $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathrm{SO}(3)$-action given above. All Σ_n^7 with odd n are equivariantly homeomorphic to the Gromoll-Meyer sphere Σ_{GM}^7 with the above $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathrm{SO}(3)$-action. If n is even all fixed point sets in Σ_n^7 are spheres while if n is odd there are also 3-dimensional fixed point sets with fundamental groups $\mathbb{Z}_2$ and $\mathbb{Z}_3$.*

The even/odd grading of the Σ_n^7 also transfers to some of the invariant submanifolds. The most important one is Σ_n^5 whose preimage under the map $E_n^{10} \to \Sigma_n^7$ consists of points (u, v) where both quaternionic components of u are purely imaginary.

Proposition 3. *Σ_n^5 is $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathrm{SO}(3)$-equivariantly diffeomorphic to $\mathbb{S}^5 \subset (\operatorname{Im}\mathbb{H})^2$ with the linear action $(B, \pm q) \cdot u = Bqu\bar{q}$ if n is even and to the Brieskorn sphere W_3^5 if n is odd. The subsphere Σ_n^5 is minimal for every $\{\pm \mathbb{1}\} \times \mathrm{SO}(3)$-invariant metric on Σ_n^7.*

Recall here that the Brieskorn sphere W_d^5 with $d \in \mathbb{N}$ is the intersection of the unit sphere in $\mathbb{C}^4 = \mathbb{C} \oplus \mathbb{C}^3$ with the complex hypersurface

$$z_0^3 + z_1^2 + z_2^2 + z_3^2 \;=\; 0$$

and that there is a natural $\mathrm{O}(2) \times \mathrm{SO}(3)$-action on W_d^5:

$$\begin{aligned} \mathrm{O}(2) \times \mathrm{SO}(3) \times W_d^5 &\to W_d^5, \\ \big(\big(\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\big), A\big) \cdot (z_0, z) \;&=\; (e^{2i\theta} z_0, e^{di\theta} A z), \\ \big(\big(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\big), A\big) \cdot (z_0, z) \;&=\; (\bar{z}_0, A\bar{z}). \end{aligned} \tag{1}$$

The classification theorems of Jänich and Hsiang-Hsiang imply that for $G = \mathrm{O}(2) \times \mathrm{SO}(3)$ and even for the smaller group $G = \{\pm \mathbb{1}\} \times \mathrm{SO}(3)$ the Brieskorn sphere W_d^5 is not G-equivariantly homeomorphic to $\mathbb{S}^5$ with any linear action, see [HMa]. However, W_d^5 is $\mathrm{SO}(3)$-equivariantly diffeomorphic to $\mathbb{S}^5$. In the case $d = 3$ an explicit formula for such a diffeomorphism is given in [DP].

The invariant subsphere Σ_n^5 is dual to the invariant circle Σ_n^1 whose preimage under the map $E_n^{10} \to \Sigma_n^7$ consists of points (u, v) for which both components of u are real. These two dual submanifolds play a central role for the geodesic geometry of Σ_n^7. We construct a one parameter family of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathrm{SO}(3)$-invariant metrics $\langle\,\cdot\,,\,\cdot\,\rangle_\nu$ on each Σ_n^7 with the following property:

Theorem 4. *All points $p \in \Sigma_n^1$ have the wiedersehen property, i.e., every unit speed geodesic γ in Σ_n^7 with $\gamma(0) = p$ is length minimizing on $[0, \pi[$ and obeys $\gamma(\pi) = -p$ and $\gamma(2\pi) = p$. Moreover, Σ_n^1 and Σ_n^5 have constant distance $\frac{\pi}{2}$ and the map $\Sigma_n^1 * \Sigma_n^5 \to \Sigma_n^7$ that maps (x, y, t) to $\gamma(t)$, where $\gamma : [0, \frac{\pi}{2}] \to \Sigma^7$ is the unique unit speed geodesic segment from x to y, is a homeomorphism.*

This invariant geodesic join structure actually is the key to prove Theorem 1 and Theorem 2. In the particular case of exotic 7-spheres our method is an improvement over the general construction that equips all exotic spheres with pointed wiedersehen metrics [Bs].

The even/odd grading of the Σ_n^7 is in contrast to what happens for the Milnor spheres $M_{k,l}^7$ and the Brieskorn spheres $W_{6n-1,3}^7$.

The Milnor sphere $M_{k,l}^7$ with $k + l = 1$ is defined by gluing two copies of $\mathbb{H} \times \mathbb{S}^3$ along $(\mathbb{H} \setminus \{0\}) \times \mathbb{S}^3$ by the map

$$(u, v) \mapsto \big(\tfrac{u}{|u|^2}, (\tfrac{u}{|u|})^k \, v \, (\tfrac{u}{|u|})^l\big). \tag{2}$$

For convenience, we set $M_{k,l}^7 = M_d^7$ where $d = k - l$ is odd. The Milnor sphere M_d^7 represents the $\frac{d^2-1}{8}$-th element in Θ_7, see [EK]. There is a natural $\mathrm{SO}(3) = \mathbb{S}^3/\{\pm 1\}$-action on M_d^7 which is in both charts defined by

$$\pm q \bullet (u, v) = (qu\bar{q}, qv\bar{q}).$$

Davis [Da] has shown that M_d^7 is $\mathrm{SO}(3)$-equivariantly diffeomorphic to $M_{d'}^7$ if and only if $d' = \pm d$ and that all M_d^7 are $\mathrm{SO}(3)$-equivariantly homeomorphic to $\mathbb{S}^7 \subset \mathbb{H}^2$

with the linear SO(3)-action given by $(\pm q, u) \mapsto qu\bar{q}$. We show that the latter situation changes when one extends the SO(3)-action by the commuting involution

$$(u, v) \mapsto (u, -v)$$

in both charts. This involution fixes all points in the base of the bundle $M_d^7 \to \mathbb{S}^4$ and induces the antipodal map on all the $\mathbb{S}^3$-fibers. For consistency, the group generated by SO(3) and the involution is denoted by $\{\pm \mathbb{1}\} \times \mathrm{SO}(3)$.

Theorem 5. *The fixed point set of the involution $(-\mathbb{1}, \pm i)$ on M_d^7 is a 3-dimensional lens space with fundamental group $\mathbb{Z}_{|d|}$. Hence, M_d^7 is $\{\pm \mathbb{1}\} \times \mathrm{SO}(3)$-equivariantly homeomorphic to $M_{d'}^7$ if and only if $d = \pm d'$. Moreover, for $|d| > 3$ none of the M_d^7 are $\{\pm \mathbb{1}\} \times \mathrm{SO}(3)$-equivariantly homeomorphic to any of the Σ_n^7.*

This theorem is a consequence of Theorem 5.1 which is the analogue of Proposition 3 for the Milnor spheres.

Grove and Ziller [GZ] constructed SO(3)-actions on M_d^7 that are entirely different from the SO(3)-actions on M_d^7 and Σ_n^7 above. The SO(3)-actions on M_d^7 and Σ_n^7 fix a circle pointwise while the Grove-Ziller actions are almost free.

The Brieskorn sphere $W_{6n-1,3}^7$ is defined by the intersection of the the unit sphere $\mathbb{S}^9 \subset \mathbb{C}^5 = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}^3$ with the complex hypersurface

$$w^{6n-1} + z_0^3 + z_1^2 + z_2^2 + z_3^2 = 0.$$

It represents the (n mod 28)-th homotopy sphere in Θ_7 (see [Bk]) and admits the natural SO(3)-action $\big(A, (w, z_0, z)\big) \mapsto (w, z_0, Az)$.

Theorem 6. *None of the $W_{6n-1,3}^7$ are SO(3)-equivariantly diffeomorphic to any of the Σ_n^7 or to any of the $M_{k,l}^7$.*

In particular, $W_{6n-1,3}^7$ is not SO(3)-equivariantly homeomorphic to the join of a circle and W_3^5. Thus, the equivariant topology of the Σ_n^7 with odd n is much more determined by the equivariant topology of W_3^5 than the equivariant topology of $W_{6n-1,3}^7$ although the latter contain W_3^5 in a much more obvious way (just by setting $w = 0$).

Many of the constructions in this paper generalize the constructions given in [DP] for the original Gromoll-Meyer sphere Σ_{GM}^7.

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2. A construction of the $\mathbb{S}^3$-principal bundles over $\mathbb{S}^7$

Recall from the introduction the definition of $E_n^{10} \subset \mathbb{S}^7 \times \mathbb{S}^7$:

$$E_n^{10} = \big\{(u, v) \in \mathbb{S}^7 \times \mathbb{S}^7 \;\big|\; \langle\!\langle \rho_n(u), v \rangle\!\rangle = 0\big\}.$$

For $n = 1$ the space E_1^{10} can be equivalently seen as the group Sp(2) of 2×2 quaternionic matrices A such that $\bar{A}^{\mathrm{t}} A = \mathbb{1}$. The standard projection $\mathrm{Sp}(2) \to \mathbb{S}^7$, $A = (u, v) \mapsto u$ turns Sp(2) into an $\mathbb{S}^3$-principal bundle over $\mathbb{S}^7$.

Lemma 2.1. *E_n^{10} is the pull-back of $\mathrm{Sp}(2)$ by the map $\rho_n : \mathbb{S}^7 \to \mathbb{S}^n$.*

Proof. By the usual explicit construction, the total space of the pull-back bundle $\rho_n^*\big(\mathrm{Sp}(2)\big)$ is the submanifold of $\mathbb{S}^7 \times \mathrm{Sp}(2)$ consisting of all pairs (u, A) such that $\rho_n(u)$ is the first colum of A. It is evident, however, that in this construction we log the first column of A twice. Eliminating this redundancy leads to the definition of E_n^{10} above. This in particular shows that E_n^{10} is a submanifold of $\mathbb{S}^7 \times \mathbb{S}^7$. □

Corollary 2.2. *E_n^{10} is an $\mathbb{S}^3$-principal bundle over $\mathbb{S}^7$ classified by $n \bmod 12$.*

Proof. The $\mathbb{S}^3$-principal bundles over $\mathbb{S}^7$ are classified by $\pi_6(\mathbb{S}^3) \approx \mathbb{Z}_{12}$ and the characteristic map of the bundle $\mathrm{Sp}(2) \to \mathbb{S}^7$ generates $\pi_6(\mathbb{S}^3)$ (see [Hu] or [DMR] for a more explicit reference). The map ρ_n has degree n. □

The principal bundle map $E_n^{10} \to \mathbb{S}^7$ is given by the projection to the first column. The corresponding free $\mathbb{S}^3$-action on E_n^{10} is given by

$$\mathbb{S}^3 \times E_n^{10} \to E_n^{10}, \quad q \bullet (u, v) = (u, v\bar{q}).$$

The map $\tilde{\rho}_n$ in the pull-back diagram

$$\begin{tikzcd} E_n^{10} \arrow[r, "\tilde{\rho}_n"] \arrow[d] & \mathrm{Sp}(2) \arrow[d] \\ \mathbb{S}^7 \arrow[r, "\rho_n"] & \mathbb{S}^7 \end{tikzcd}$$

takes the explicit form

$$\tilde{\rho}_n : E_n^{10} \to \mathrm{Sp}(2), \quad (u, v) \mapsto (\rho_n(u), v).$$

Recall from the introduction that there is a free $\mathbb{S}^3$-action $q \star (u, v) = (qu\bar{q}, qv)$ on E_n^{10} that commutes with the $\bullet$-action and whose orbit space is the smooth manifold Σ_n^7. The pull-back diagram above extends to the following commutative diagram:

$$\begin{tikzcd} E_n^{10} \arrow[rr, "\tilde{\rho}_n"] \arrow[dr] \arrow[dd] & & \mathrm{Sp}(2) \arrow[dr] \arrow[dd] & \\ & \Sigma_n^7 \arrow[rr, "\rho_n'"] & & \Sigma_{\mathrm{GM}}^7 \\ \mathbb{S}^7 \arrow[rr, "\rho_n"] & & \mathbb{S}^7 & \end{tikzcd}$$

The degree of the induced map $\rho_n' : \Sigma_n^7 \to \Sigma_{\mathrm{GM}}^7$ is n. The proof that Σ_n^7 represents the ($n \bmod 28$)-th element of Θ_7 requires several geometric constructions and is postponed until section 4.

Each principal bundle E_n^{10} admits a natural action of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3$, where $\mathbb{Z}_2 \times \mathbb{Z}_2$ denotes the diagonal matrices in $\mathrm{O}(2) \subset \mathrm{Sp}(2)$:

$$\mathbb{Z}_2 \times \mathbb{Z}_2 \times E_n^{10} \to E_n^{10}, \quad B \cdot (u, v) = (Bu, Bv), \tag{3}$$

$$\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3 \times E_n^{10} \to E_n^{10}, \quad (q_1, q_2, q_3) \cdot \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix} = \begin{pmatrix} q_1 u_1 \bar{q}_1 & q_1 v_1 \bar{q}_3 \\ q_2 u_2 \bar{q}_1 & q_2 v_2 \bar{q}_3 \end{pmatrix}. \tag{4}$$

Lemma 2.3. *This action on E_n^{10} is of cohomogeneity 2.*

Proof. The third $\mathbb{S}^3$-factor yields the principal action related to the bundle $E^{10}_n \to \mathbb{S}^7$, $(u,v) \mapsto u$, i.e., this $\mathbb{S}^3$-factor acts simply transitively on the fiber over any $u \in \mathbb{S}^7$. The action of the first two $\mathbb{S}^3$-factors on $\mathbb{S}^7$ has kernel $\{\pm(1,1)\}$ and induces a standard linear SO(4)-action on $\mathbb{S}^7$. By applying all three $\mathbb{S}^3$-factors one can transform an arbitrary point in E^{10}_n to a point of the form

$$\left(\begin{smallmatrix} \cos t + i\cos s \sin t & -\sin s \sin nt \\ \sin s \sin t & \cos nt - i \cos s \sin nt \end{smallmatrix}\right). \qquad \square$$

The diagonal in the first two $\mathbb{S}^3$-factors gives the Gromoll-Meyer action $\star$ corresponding to the principal bundle $E^{10}_n \to \Sigma^7_n$. The third $\mathbb{S}^3$-factor and the $\mathbb{Z}_2 \times \mathbb{Z}_2$-factor yield the effective $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathrm{SO}(3)$-action $\bullet$ on Σ^7_n from the introduction. It is an interesting question for which n this $\bullet$-action can be extended. The maximum dimension of any compact differentiable transformation group of an exotic 7-sphere is four [St]. On the original Gromoll-Meyer sphere $\Sigma^7_{\mathrm{GM}} = \Sigma^7_1$ there is a natural $\mathrm{O}(2) \times \mathrm{SO}(3)$-action. It is induced by the action

$$\mathrm{O}(2) \times \mathrm{SO}(3) \times \mathrm{Sp}(2) \to \mathrm{Sp}(2), \quad (A,q) \bullet (u,v) \mapsto (Au, Av\bar{q})$$

on $\mathrm{Sp}(2) = E^{10}_1$ and extends the $\bullet$-action naturally. A corresponding $\mathrm{O}(2) \times \mathrm{SO}(3)$-action exists of course on Σ^7_{-1}. On Σ^7_0 an $\mathrm{O}(2) \times \mathrm{SO}(3)$-action is induced by the action

$$\mathrm{O}(2) \times \mathrm{SO}(3) \times E^{10}_0 \to E^{10}_0, \quad (A,q) \bullet (u,v) \mapsto (Au, v\bar{q}).$$

On the other Σ^7_n with $n \neq -1, 0, 1$, however, it seems likely that the $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathrm{SO}(3)$-action cannot be extended to any larger group, see Remark 4.5.

Question 2.4. Which E^{10}_n admit Riemannian metrics with $K \geq 0$ that are invariant under the cohomogeneity 2 action above? If some E^{10}_n admits such a metric then by the O'Neill formulas the induced metric on Σ^7_n also has $K \geq 0$. This would be particularly interesting for those Σ^7_n that are not diffeomorphic to $\mathbb{S}^3$-bundles over $\mathbb{S}^4$ since on such exotic spheres no metrics with $K \geq 0$ are known so far.

Remark 2.5. While there are twelve $\mathbb{S}^3$-principal bundles over $\mathbb{S}^7$ there are 28 homotopy 7-spheres. This means in particular that some Σ^7_n are quotients of trivial bundles E^{10}_n. This phenomenon is well-known from surgery theory (see [Wa]) in an inexplicit way.

Remark 2.6. Grove and Ziller [GZ] constructed cohomogeneity one metrics with $K \geq 0$ on all $\mathbb{S}^3 \times \mathbb{S}^3$-principal bundles over $\mathbb{S}^4$. It is known that the (n mod 12)-th $\mathbb{S}^3$-principal bundle over $\mathbb{S}^4$ is diffeomorphic to an $\mathbb{S}^3 \times \mathbb{S}^3$-principal bundles over $\mathbb{S}^4$ if and only if $n \bmod 12 \in \{0,1,3,4,6,7,9,10\}$. It is easy to see that the set of all integers n with $n \bmod 12 \in \{0,1,3,4,6,7,9,10\}$ maps surjectively on $\mathbb{Z}_{28}$. Thus, every element in Θ_7 can be represented by some Σ^7_n such that E^{10}_n admits a cohomogeneity one metric with $K \geq 0$. However, this does not mean that Σ^7_n admits a metric with $K \geq 0$ since the Gromoll-Meyer action E^{10}_n is not isometric with respect to the Grove-Ziller metric.

3. Invariant submanifolds and parity

In this section we will see that the even/odd grading of the generalized Gromoll-Meyer spheres Σ^7_n is based on an elementary property of the maps ρ_n.

Consider the subsets

$$E^9_n := \{(u,v) \in E^{10}_n \mid u \in \operatorname{Im}\mathbb{H} \times \mathbb{H}\},$$
$$E^8_n := \{(u,v) \in E^{10}_n \mid u \in \operatorname{Im}\mathbb{H} \times \operatorname{Im}\mathbb{H}\}$$

of $E^{10}_n \subset \mathbb{S}^7 \times \mathbb{S}^7$. These are the preimages of the subspheres

$$\mathbb{S}^6 = \big\{ \left(\begin{smallmatrix} p \\ w \end{smallmatrix}\right) \mid p \in \operatorname{Im}\mathbb{H},\ w \in \mathbb{H},\ |p|^2 + |w|^2 = 1 \big\},$$
$$\mathbb{S}^5 = \big\{ \left(\begin{smallmatrix} p_1 \\ p_2 \end{smallmatrix}\right) \mid p_1, p_2 \in \operatorname{Im}\mathbb{H},\ |p_1|^2 + |p_2|^2 = 1 \big\}$$

of $\mathbb{S}^7 \subset \mathbb{H} \times \mathbb{H}$ under the principal bundle projection $E^{10}_n \to \mathbb{S}^7$.

Lemma 3.1. *E^9_n and E^8_n are submanifolds of E^{10}_n diffeomorphic to $\mathbb{S}^6 \times \mathbb{S}^3$ and $\mathbb{S}^5 \times \mathbb{S}^3$, respectively.*

Proof. $E^9_n \to \mathbb{S}^6$ is a proper subbundle of $E^{10}_n \to \mathbb{S}^7$ and hence trivial. □

Lemma 3.2. *E^9_n and E^8_n are invariant under the free $\star$-action of $\mathbb{S}^3$ and under the $\bullet$-action of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{S}^3$. Hence, the $\star$-quotients Σ^6_n and Σ^5_n are submanifolds of Σ^7_n with a natural $\bullet$-action of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathrm{SO}(3)$.*

Proof. Straightforward. □

Lemma 3.3. *As submanifolds of $\mathbb{S}^7 \times \mathbb{S}^7$ we have*

$$\ldots = E^9_{-3} = E^9_{-1} = E^9_1 = E^9_3 = \ldots,$$
$$\ldots = E^9_{-4} = E^9_{-2} = E^9_0 = E^9_2 = E^9_4 = \ldots$$

and the same identities also hold for $E^8_n \subset E^9_n$ and for the quotients Σ^6_n and Σ^5_n.

Proof. This is an immediate consequence of the two basic identities

$$\rho_{2m+1}\big(\left(\begin{smallmatrix} p \\ w \end{smallmatrix}\right)\big) = (-1)^m \left(\begin{smallmatrix} p \\ w \end{smallmatrix}\right) \quad \text{and} \quad \rho_{2m}\big(\left(\begin{smallmatrix} p \\ w \end{smallmatrix}\right)\big) = (-1)^m \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$$

for $\left(\begin{smallmatrix} p \\ w \end{smallmatrix}\right) \in \mathbb{S}^6 \subset \operatorname{Im}\mathbb{H} \times \mathbb{H}$. □

Corollary 3.4. *If n is odd, Σ^5_n is equivariantly diffeomorphic to the Brieskorn sphere W^5_3 with its natural $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathrm{SO}(3)$-action. If n is even, Σ^5_n is equivariantly diffeomorphic to the Euclidean sphere $\mathbb{S}^5 \subset \mathbb{R}^3 \times \mathbb{R}^3$ where $\mathrm{SO}(3)$-acts diagonally on both $\mathbb{R}^3$-factors and each $\mathbb{Z}_2$-factor acts on one of the $\mathbb{R}^3$-factors.*

Proof. From Lemma 7.4 of [DP] we know that Σ^5_1 is $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathrm{SO}(3)$-equivariantly diffeomorphic to W^5_3. For Σ^5_0 we observe that

$$E^8_0 = \big\{ \left(\begin{smallmatrix} p_1 & 0 \\ p_2 & q \end{smallmatrix}\right) \mid p_1, p_2 \in \operatorname{Im}\mathbb{H}, q \in \mathbb{S}^3 \big\}$$

The natural embedding $\mathbb{S}^5 \to E^8_0$, $\left(\begin{smallmatrix} p_1 \\ p_2 \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} p_1 & 0 \\ p_2 & 1 \end{smallmatrix}\right)$ identifies the $\star$-quotient of E^8_0 with $\mathbb{S}^5$. □

Lemma 3.5. *The subsphere Σ^5_n is minimal in Σ^6_n and Σ^7_n for all $\{\pm\mathbb{1}\} \times \mathrm{SO}(3)$-invariant Riemannian metrics on Σ^7_n.*

Proof. Analogous to the proof of Corollary 3.4 in [DP] this follows from the fact that Σ^5_n is the union of orbits whose isotropy groups contain elements of the form $(-\mathbb{1}, \pm q)$. □

4. The geodesic join structure of Σ^7_n

We will now study the geometry of a one parameter family of Riemannian metrics on E^{10}_n and Σ^7_n and use the results to prove Theorem 1, Theorem 2 and Theorem 4. The one parameter family of metrics is defined in such a way that the constructions of [Du] and [DP] for Σ^7_{GM} can be extended to all Σ^7_n.

We equip the total space of the principal bundle $E^{10}_n \to \mathbb{S}^7$ with the Riemannian metric $\langle\,\cdot\,,\,\cdot\,\rangle_\nu$ with $\nu > 0$ defined by the following properties:

- The $\mathbb{S}^3$-fibers have constant curvature $\frac{1}{\nu}$.
- The horizontal distribution is given by the pull-back of the horizontal distribution of $\mathrm{Sp}(2)$ via the map ρ_n, i.e., we pull-back the principal bundle connection of $\mathrm{Sp}(2)$.
- The metric $\langle\,\cdot\,,\,\cdot\,\rangle_\nu$ induces on $\mathbb{S}^7$ the metric with constant curvature 1 by Riemannian submersion.

Such metrics are called connection metrics or Kaluza-Klein metrics.

The $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^3$-action given in (3) and (4) is isometric with respect to the metric $\langle\,\cdot\,,\,\cdot\,\rangle_\nu$. In particular, the Gromoll-Meyer action $\star$ is isometric and Σ^7_n inherits a Riemannian metric by Riemannian submersion, which will again be denoted by $\langle\,\cdot\,,\,\cdot\,\rangle_\nu$. The $\bullet$-action of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{S}^3$ on E^{10}_n is also isometric. Since the $\bullet$-action commutes with the $\star$-action, it induces an effective isometric $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathrm{SO}(3)$-action on $(\Sigma^7_n, \langle\,\cdot\,,\,\cdot\,\rangle_\nu)$.

Lemma 4.1. *The common fixed point set of* $\mathrm{SO}(3)$ *in* Σ^7_n *is the circle*

$$\Sigma^1_n := \left\{\pi_{\Sigma^7_n}\left(\left(\begin{smallmatrix}\cos t & -\sin nt\\ \sin t & \cos nt\end{smallmatrix}\right)\right) \mid t \in \mathbb{R}\right\}.$$

Hence, for any $\mathrm{SO}(3)$*-invariant Riemannian metric on* Σ^7_n*, this circle* Σ^1_n *is a simple closed geodesic.*

Proof. $\pi_{\Sigma^7_n}(u,v)$ is a fixed point of $\mathrm{SO}(3)$ if and only if for every $q \in \mathbb{S}^3$ there is a $q' \in \mathbb{S}^3$ such that $(q'u\bar{q}', q'v\bar{q}) = (u,v)$. It is easy to see from the second column of this equation that all elements of $\mathbb{S}^3$ occur for q'. Therefore, u must have two real components. □

Note that the $\mathbb{Z}_2 \times \mathbb{Z}_2$-action on Σ^1_n is equivalent to the standard $\mathbb{Z}_2 \times \mathbb{Z}_2$-action on $\mathbb{S}^1$. In particular, for each point $p \in \Sigma^1_n$ there is a natural antipode $-p$.

Theorem 4.2. *Every unit speed geodesic* γ *in* $(\Sigma^7_n, \langle\,\cdot\,,\,\cdot\,\rangle_\nu)$ *with* $\gamma(0) = p \in \Sigma^1_n$ *is length minimizing on* $[0,\pi[$ *and we have* $\gamma(\pi) = -p$ *and* $\gamma(2\pi) = p$.

Proof. The proof is similar to the proofs of Theorem I in [Du] and Theorem 2.1 in [DP]. We lift γ horizontally to a geodesic $\tilde{\gamma}$ in E^{10}_n with

$$\tilde{\gamma}(0) = \alpha(t) := \left(\begin{smallmatrix}\cos t & -\sin nt\\ \sin t & \cos nt\end{smallmatrix}\right) \in E^1_n.$$

That $\tilde{\gamma}$ is horizontal with respect to $E^{10}_n \to \Sigma^7_n$ means that the geodesic $\tilde{\gamma}$ passes perpendicularly through all $\star$-orbits. It is straightforward to check that

$$\mathbb{S}^3 \star \alpha(t) = \mathbb{S}^3 \bullet \alpha(t).$$

Thus, $\tilde{\gamma}$ passes perpendicularly through $\mathbb{S}^3 \bullet \tilde{\gamma}(0)$. A geodesic that passes perpendicularly through one orbit passes perpendicularly through all orbits. Hence, $\tilde{\gamma}$ passes perpendicularly through all $\mathbb{S}^3$-orbits of the $\bullet$-action. In other words, $\tilde{\gamma}$ is horizontal to the principal fibration $E^{10}_n \to \mathbb{S}^7$. Hence, $\tilde{\gamma}$ projects to a geodesic β in $\mathbb{S}^7$. By definition of $\langle\,\cdot\,,\,\cdot\,\rangle_\nu$ the sphere $\mathbb{S}^7$ inherits the metric with constant curvature 1 from E^{10}_n by Riemannian submersion. Since all unit speed geodesics of $\mathbb{S}^7$ that start at $\beta(0) = \pi_{\mathbb{S}^7}(\alpha(t))$ pass through $\beta(\pi) = -\beta(0)$ at time π we have $\beta(\pi) = \pi_{\mathbb{S}^7}(\alpha(t+\pi))$. Thus, $\tilde{\gamma}(\pi)$ is contained in $\mathbb{S}^3 \bullet \alpha(t+\pi) = \mathbb{S}^3 \star \alpha(t+\pi)$ and $\tilde{\gamma}(2\pi)$ is contained in $\mathbb{S}^3 \star \alpha(t+2\pi) = \mathbb{S}^3 \bullet \tilde{\gamma}(0)$. This shows $\gamma(\pi) = -\gamma(0)$ and $\gamma(2\pi) = \gamma(0)$. Now let γ be a unit speed geodesic in Σ^7_n with $\gamma(0) = p$ and $\gamma_1(l) = -p$. By the construction above β is a unit speed geodesic in $\mathbb{S}^7$ with $\beta(l) = -\beta(0)$. Hence, l cannot be less than π. $\square$

Recall that the join $X{*}Y$ of two spaces X and Y is the quotient of $X\times Y\times[0,1]/\sim$ where $(x,y,0)\sim(x,y',0)$ and $(x,y,1)\sim(x',y,1)$ for all $x\in X$ and all $y\in Y$. For our purposes it is convenient to substitute $[0,1]$ by $[0,\frac{\pi}{2}]$.

Corollary 4.3. *Σ^1_n and Σ^5_n have constant distance $\frac{\pi}{2}$. Moreover, the map $\Sigma^1_n{*}\Sigma^5_n \to \Sigma^7_n$ that maps (x,y,t) to $\gamma(t)$, where $\gamma : [0,\frac{\pi}{2}] \to \Sigma^7$ is the unique unit speed geodesic segment from x to y, is an equivariant homeomorphism.*

Proof. This follows from the construction in the proof of Theorem 4.2 if one recalls that the submanifolds E^1_n and E^9_n of E^{10}_n project to the submanifolds

$$\mathbb{S}^1 = \left\{ \left(\begin{smallmatrix}\cos t\\ \sin t\end{smallmatrix}\right) \,\middle|\, t\in\mathbb{R}\right\},$$
$$\mathbb{S}^5 = \left\{ \left(\begin{smallmatrix}p_1\\ p_2\end{smallmatrix}\right) \,\middle|\, p_1,p_2\in\operatorname{Im}\mathbb{H},\ |p_1|^2+|p_2|^2=1\right\}$$

of $\mathbb{S}^7\subset\mathbb{H}^2$ under the principal fibration $E^{10}_n\to\mathbb{S}^7$ and to the submanifolds Σ^1_n and Σ^5_n of Σ^7_n under the principal fibration $E^{10}_n\to\Sigma^7_n$. $\square$

Theorem 4.2 and Corollary 4.3 together yield Theorem 4 from the introduction.

Corollary 4.4. *Σ^7_n is $\mathbb{Z}_2\times\mathbb{Z}_2\times\mathrm{SO}(3)$-equivariantly homeomorphic to $\mathbb{S}^1*\mathbb{S}^5$ if n is even and to $\mathbb{S}^1*W^5_3$ if n is odd. Here, the $\mathbb{Z}_2\times\mathbb{Z}_2$ acts on $\mathbb{S}^1$ in the standard way.*

Proof. This is evident from Corollary 3.4 and Corollary 4.3. $\square$

In particular, all Σ^7_n with even n are mutually equivariantly homeomorphic and that all Σ^7_n with odd n are mutually equivariantly homeomorphic. This proves Theorem 2 from the introduction.

Proof of Theorem 1. Consider the unit speed geodesic

$$\beta(t) = \left(\begin{smallmatrix}\cos t+p\sin t\\ w\sin t\end{smallmatrix}\right)$$

in $\mathbb{S}^7 \subset \mathbb{H}^2$ that emanates from the north pole with initial velocity $\left(\begin{smallmatrix} p \\ w \end{smallmatrix}\right) \in \mathbb{S}^6 \subset \operatorname{Im}\mathbb{H} \times \mathbb{H}$. A lift $\tilde{\gamma}_n$ of this curve to E_n^{10} with $\tilde{\gamma}_n(0) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ is given by

$$\tilde{\gamma}_n(t) = \begin{pmatrix} \cos t + p \sin t & -e^{ntp}\bar{w}\sin(nt) \\ w \sin t & \frac{w}{|w|} e^{ntp}(\cos(nt) - p\sin(nt))\frac{\bar{w}}{|w|} \end{pmatrix}. \tag{5}$$

Here $e^p = \cos|p| + \frac{p}{|p|}\sin|p|$ denotes the exponential map of $\mathbb{S}^3 \subset \mathbb{H}$ at 1. Note that for $w = 0$ equation (5) simply becomes $\tilde{\gamma}_n(t) = \left(\begin{smallmatrix} e^{ntp} & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Using the identity

$$\tilde{\rho}_n(\tilde{\gamma}_n(t)) = \tilde{\gamma}_1(nt)$$

for the map $\tilde{\rho}_n : E_n^{10} \to \mathrm{Sp}(2)$ defined in section 2 it is straightforward to verify that $\tilde{\gamma}_n$ is the unique *horizontal* lift of β to E_n^{10} with $\tilde{\gamma}_n(0) = \mathbb{1}$. Since the fibers of $E_n^{10} \to \mathbb{S}^7$ and $E_n^{10} \to \Sigma_n^7$ through $\tilde{\gamma}_n(0) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ are the same (as sets), the geodesic $\tilde{\gamma}_n$ is horizontal with respect to both these fibrations. This shows that $\gamma_n = \pi_{\Sigma_n^7} \circ \tilde{\gamma}_1$ is a geodesic in Σ_n^7. Now, considering all possible unit initial vectors $\left(\begin{smallmatrix} p \\ w \end{smallmatrix}\right) \in \mathbb{S}^6 \subset \operatorname{Im}\mathbb{H} \times \mathbb{H}$ and times $t \in [0, \frac{\pi}{2}]$ the geodesics γ_n provide an embedding of a disk $D^7(\frac{\pi}{2})$ into Σ_n^7 by Theorem 4.2. In the same way, the geodesics $\pi_{\Sigma^7} \circ (-\tilde{\gamma}_n) \circ (-\mathrm{id})$ provide another embedding of the same disk. By Theorem 4.2, Σ_n^7 is the twisted sphere obtained by gluing these two embedded disks along their common boundary. One easily checks that

$$\tilde{\gamma}_n(p, w, \tfrac{\pi}{2}) = q \star (-\tilde{\gamma}_n(-p', -w', \tfrac{\pi}{2}))$$

for some $q \in \mathbb{S}^3$ if and only if $(p', w') = \sigma^n(p, w)$ where σ is the exotic diffeomorphism of $\mathbb{S}^6 \subset \operatorname{Im}\mathbb{H} \times \mathbb{H}$ first described in [Du]. This diffeomorphism σ generates $\pi_0(\mathrm{Diff}_+(\mathbb{S}^6))$. It is given by the formula

$$\sigma(p, w) := \overline{\mathrm{b}(p, w)}(p, w)\mathrm{b}(p, w)$$

where $\mathrm{b}(p, w) = \frac{w}{|w|} e^{\pi p} \frac{\bar{w}}{|w|}$ is an analytic formula for a generator of $\pi_6(\mathbb{S}^3)$. Hence, we have obtained Σ_n^7 by gluing two 7-disks with the n-th power of a generator of $\pi_0(\mathrm{Diff}_+(\mathbb{S}^6)) \approx \Theta_7 \approx \mathbb{Z}_{28}$. □

Remark 4.5. Let G be a compact group acting smoothly on Σ_n^7 with $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathrm{SO}(3) \subset G$. Precisely as in [DP], Lemma 3.7, one can show that G leaves Σ_n^1 and Σ_n^5 invariant. Let $n \notin \{-1, 0, 1\}$. Comparing for different $p \in \Sigma_n^1$ the closing behaviour of geodesics that start at p perpendicularly to Σ_n^1, one can see that $\mathbb{Z}_2 \times \mathbb{Z}_2$ is the maximal compact group that acts isometrically on $(\Sigma_n^7, \langle\,\cdot\,,\,\cdot\,\rangle_\nu)$ and effectively on the circle Σ_n^1. This difference from the cases $n = -1, 0, 1$ suggests that $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathrm{SO}(3)$ is the full isometry group of $(\Sigma_n^7, \langle\,\cdot\,,\,\cdot\,\rangle_\nu)$.

Remark 4.6. If we pull back the metric $\langle\,\cdot\,,\,\cdot\,\rangle_\nu$ on $\mathrm{Sp}(2)$ by the map $\tilde{\rho}_n$ then we obtain a degenerate metric $\langle\,\cdot\,,\,\cdot\,\rangle'_\nu$ on E_n^{10} that has the same geodesics through the circle Σ_n^1 as the metric $\langle\,\cdot\,,\,\cdot\,\rangle_\nu$. For $n \notin \{-1, 0, 1\}$ the metric $\langle\,\cdot\,,\,\cdot\,\rangle'_\nu$ is degenerate precisely over $|n| - 1$ subspheres in $\mathbb{S}^7$ whose first quaternionic components have constant real part. With such a metric Σ_n^7 looks like n copies of Σ_{GM}^7 stacked one on top of the other, i.e., like a degenerate connected sum of n copies of Σ_{GM}^7.

Remark 4.7. The manifolds $(E^9_n, \langle\,\cdot\,,\,\cdot\,\rangle_\nu)$ with even n are not just mutually equal as submanifolds of $\mathbb{S}^7 \times \mathbb{S}^7$ but also mutually equal as Riemannian manifolds. Hence, also the manifolds $(\Sigma^6_n, \langle\,\cdot\,,\,\cdot\,\rangle_\nu)$ with even n are all mutually equal as Riemannian manifolds. The analogous statements hold for odd n.

5. Comparison to the exotic Milnor and Brieskorn 7-spheres

In this section we compare the equivariant topology of the spheres Σ^7_n with the equivariant topology of the Milnor spheres M^7_d and the Brieskorn spheres $W^7_{6n-1,3}$ and prove Theorem 5 and Theorem 6 of the introduction.

Recall from the introduction that the Milnor spheres M^7_d admit natural $\{\pm\mathbb{1}\} \times \mathrm{SO}(3)$-actions. Davis [Da] has shown that these actions can be extendend to $\mathrm{GL}(2,\mathbb{R}) \times \mathrm{SO}(3)$-actions. In the first chart the $\mathrm{GL}(2,\mathbb{R})$-action is given by

$$\left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right) \bullet (u,v) = \Big(\tfrac{au+c}{bu+d}, \det\left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right)\left(\tfrac{bu+d}{|bu+d|}\right)^k v \left(\tfrac{bu+d}{|bu+d|}\right)^l\Big)$$

and in the second one by

$$\left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right) \bullet (u,v) = \Big(\tfrac{b+du}{a+cu}, \det\left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right)\left(\tfrac{a+c\bar{u}}{|a+c\bar{u}|}\right)^k v \left(\tfrac{a+c\bar{u}}{|a+c\bar{u}|}\right)^l\Big).$$

Note that our definition of the action differs from the definition given by Davis by the factor $\det\left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right)$. The reason is that with our definition the identification between M^7_3 and the Gromoll-Meyer sphere Σ^7_{GM} given in [GM] becomes an $\mathrm{O}(2) \times \mathrm{SO}(3)$-equivariant diffeomorphism while without the determiant factor the identification is only $\mathrm{SO}(2)\times\mathrm{SO}(3)$-equivariant. Moreover, note that the map $M^7_d \to M^7_{-d}$ given by $(u,v) \mapsto (\bar{u},\bar{v})$ in both charts is an $\mathrm{GL}(2,\mathbb{R}) \times \mathrm{SO}(3)$-equivariant diffeomorphism.

Theorem 5.1. *In every Milnor sphere M^7_d there is a unique invariant submanifold M^5_d which is $\mathrm{O}(2)\times\mathrm{SO}(3)$-equivariantly diffeomorphic to the Brieskorn sphere $W^5_{|d|}$ with the $\mathrm{O}(2) \times \mathrm{SO}(3)$-action given in (1). This submanifold M^5_d is minimal for any $\{\pm\mathbb{1}\} \times \mathrm{SO}(3)$-invariant Riemannian metric on M^7_d.*

Proof. It suffices to consider the case $d > 0$. Let M^5_d be the submanifold of M^7_d given by the equations $\mathrm{Re}\, v = 0$ and $\mathrm{Re}\, uv = 0$ n both charts (it is essential here that $k + l = 1$). Hirsch and Milnor [HMi] proved that M^5_d is homeomorphic and hence (because exotic spheres do not exist in dimension 5) diffeomorphic to $\mathbb{S}^5$. It is straightforward to check that M^5_d is invariant under the $\mathrm{SO}(2) \times \mathrm{SO}(3)$-action. Consider the curve α in M^5_d which is given by $\alpha(s) = (i\tan s, j)$ in the first chart. The isotropy groups along α are

$$K_- = \{(\mathbb{1}, \pm e^{j\tau})\} \cup \{(-\mathbb{1}, \pm i e^{j\tau})\} \cup \big\{\big(\left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right], \pm e^{j\tau}\big)\big\} \cup \big\{\big(\left[\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right], \pm i e^{j\tau}\big)\big\}$$

at $s = 0$,

$$H = \big\{(\mathbb{1}, \pm 1), (-\mathbb{1}, \pm i), \big(\left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right], \pm j\big), \big(\left[\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right], \pm k\big)\big\}$$

for $0 < s < \frac{\pi}{4}$, and

$$K_+ = \big\{\big(\left[\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right], \pm e^{-\frac{d}{2}i\theta}\big)\big\} \cup \big\{\big(\left[\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right], \pm e^{-\frac{d}{2}i\theta} j\big)\big\}$$

at $s = \frac{\pi}{4}$. Now consider the Brieskorn sphere W^5_d with the $\mathrm{O}(2) \times \mathrm{SO}(3)$-action given in (1) and the curve

$$\beta(s) = \Big(s, 0, \tfrac{1}{\sqrt{2}}\sqrt{1 - s^2 - s^d}, -\tfrac{i}{\sqrt{2}}\sqrt{1 - s^2 + s^d}\Big)$$

on the interval $[s_-, 0]$ where $s_- < 0$ is the root of $1 - s^2 + s^d$. Straightforward computations show that the isotropy groups along β are the same as the isotropy groups along α. This proves that M^5_d and W^5_d are equivariantly diffeomorphic. The uniqueness and minimality of M^5_d follows from the following fact: The fixed point set of any element of the form $(-\mathbb{1}, \pm q)$ is contained in M^5_d and even more M^5_d can be seen to be the union of orbits whose isotropy groups contains such elements. $\square$

Proof of Theorem 5. The involution $(-\mathbb{1}, \pm i)$ is contained in $M^5_d \approx W^5_{|d|}$. The fixed point set of $(-\mathbb{1}, \pm i) = (-\mathbb{1}, \mathrm{diag}(1, -1, -1)$ in $W^5_{|d|}$ is the $W^3_{|d|}$ given by the equation $z_1 = 0$ and hence diffeomorphic to a lens space with fundamental group $\mathbb{Z}_{|d|}$. $\square$

The Milnor sphere M^7_d have direct analogues M^{15}_d in dimension 15. They are obtained by gluing two copies of $\mathbb{O} \times \mathbb{S}^7$ along $(\mathbb{O} \setminus \{0\}) \times \mathbb{S}^7$ by the map (2). Precisely as above each M^{15}_d admits a smooth action of $\mathrm{O}(2) \times \mathrm{G}_2$ (see [Da]).

Theorem 5.2. *In every M^{15}_d there is a unique invariant submanifold M^{13}_d which is $\mathrm{O}(2) \times \mathrm{G}_2$-equivariantly diffeomorphic to the Brieskorn sphere $W^{13}_{|d|}$ with the action of $\mathrm{O}(2) \times \mathrm{G}_2 \subset \mathrm{O}(2) \times \mathrm{SO}(7)$ given analogously to (1). This submanifold M^{13}_d is minimal for any $\{\pm\mathbb{1}\} \times \mathrm{G}_2$-invariant Riemannian metric on M^{15}_d.*

Proof. Analogous to the proof of Theorem 5.1. $\square$

Finally, we turn to the Brieskorn spheres $W^7_{6n-1,3}$ and prove Theorem 6 from the introduction.

Proof of Theorem 6. The involution $\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{smallmatrix}\right) \in \mathrm{SO}(3)$ on $W^7_{6n-1,3}$ is given by

$$(w, z_0, z_1, z_2, z_3) \mapsto (w, z_0, z_1, -z_2, -z_3).$$

Its fixed point set is thus identical to $W^3_{6n-1,3,2}$, which is the intersection of the unit sphere $\mathbb{S}^5$ in $\mathbb{C}^3$ with the complex hypersurface

$$w^{6n-1} + z_0^3 + z_1^2 = 0.$$

Milnor [Mi2] has shown that $W^3_{5,3,2}$ is diffeomorphic to Poincare dodecahedral space and that the universal covering space of $W^3_{6n-1,3,2}$ is non-compact if $n > 1$. $\square$

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